Induction
More Examples

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In this lecture, we see more examples of mathematical induction (section 4.1 of Rosen).

1 Recap

A simple proof by induction has the following outline:

> Proof: We will show $P(n)$ is true for all n, using induction on n.
>
> **Base**: We need to show that $P(1)$ is true.
>
> **Induction**: Suppose that $P(k)$ is true, for some integer k. We need to show that $P(k+1)$ is true.

In constructing an induction proof, you've got two tasks. First, you need to set up this outline for your problem. This includes identifying a suitable proposition P and a suitable integer variable n.

Your second task is to fill in the middle part of the induction step. That is, you must figure out how to relate a solution for a larger problem $P(k+1)$ to a solution for a small problem $P(k)$. Most students want to do this by starting with the small problem and adding something to it. For more complex situations, it's usually better to start with the larger problem and try to find an instance of the smaller problem inside it.

In the examples we saw last Friday, the claim involved a numerical formula containing only one (integer) variable. Today, we'll see some examples involving more than one variable, as well as some examples where the claim is about objects other than numbers.

2 A claim with more than one variable

Consider this claim:

Claim 1 *For any non-negative integer m and any non-negative real number x, $(1+x)^m \geq 1+mx$.*

This claim contains two variables, so it's important to be clear about which is the induction variable. In this case, only m will work because it's the only integer. You can't do induction on real numbers.

So $P(m)$ is "for any non-negative real number x, $(1+x)^m \geq 1+mx$."

> Proof: by induction on m.
>
> Base: $m = 0$. Then $(1+x)^m = (1+x)^0 = 1 = 1+0x = 1+mx$.
>
> Induction: Suppose that there is a non-negative integer k, such that $(1+x)^k \geq 1+kx$ for any non-negative real number x. We need to show that $(1+x)^{k+1} \geq 1+(k+1)x$, for any for any non-negative real number x.
>
> $(1+x)^{k+1} = (1+x)^k \cdot (1+x)$. By the induction hypothesis, $(1+x)^k \geq 1+kx$. So we have:
>
> $$(1+x)^k \cdot (1+x) \geq (1+kx) \cdot (1+x) = 1+kx+x+kx^2 = 1+(1+k)x+kx^2$$
>
> Since $x^2 \geq 0$ and k was specified to be non-negative, $1+(1+k)x+kx^2 \geq 1+(1+k)x$. So $(1+x)^{k+1} \geq 1+(k+1)x$, which is what we needed to show.

3 Induction on the size of sets

Now, let's consider a fact about sets which we've used already but never properly proved:

Claim 2 *For any finite set S containing n elements, S has 2^n subsets.*

The objects involved in this claim are sets. To apply induction to facts that aren't about the integers, we need to find a way to use the integers to organize our objects. In this case, we'll organize our sets by their cardinality.

The proposition $P(n)$ for our induction is then "For any set S containing n elements, S has 2^n subsets." Notice that each $P(k)$ is a claim about a whole family of sets, e.g. $P(1)$ is a claim about $\{37\}$, $\{\text{fred}\}$, $\{-31.7\}$, and so forth.

> Proof: We'll prove this for all sets S, by induction on the cardinality of the set.
>
> Base: Suppose that S is a set that contain no elements. Then S is the empty set, which has one subset, i.e. itself. Putting zero into our formula, we get $2^0 = 1$ which is correct.
>
> Induction: Suppose that our claim is true for all sets of k elements, where k is some non-negative integer. We need to show that it is true for all sets of $k + 1$ elements.
>
> Suppose that S is a set containing $k + 1$ elements. Since k is non-negative, $k + 1 \geq 1$, so S must contain at least one element. Let's pick a random element a in S. Let $T = S - \{a\}$.
>
> If B is a subset of S, either B contains a or B doesn't contain a. The subsets of S not containing a are exactly the subsets of T. The subsets of S containing a are exactly the subsets of T, with a added to them. So S has twice as many subsets as T.
>
> By the induction hypothesis, T has 2^k subsets. So S has $2 \cdot 2^k = 2^{k+1}$ subsets, which is what we needed to show.

Notice that, in the inductive step, we need to show that our claim is true for **all** sets of $k+1$ elements. Because we are proving a universal statement, we need to pick a representative element of the right type. This is the set S that we choose in the second paragraph of the inductive step.

4 A proof involving functions

Consider this claim:

Claim 3 *For any non-empty sets A and B where $|A| = |B| = n$, there are exactly $n!$ bijections from A to B.*

[Explicitly show all 6 bijections between two specific sets of 3 elements.]

Let's prove this using induction. Again, each $P(n)$ will be a claim about a whole collection of sets A and B.

> Proof: By induction on n, i.e. the cardinality of A and B.
>
> Base: $n = 1$. Then there is exactly one bijection from A to B, mapping the single element of A to the single element of B. In this case $n! = 1$ as well.
>
> Induction: For some positive integer k, suppose that for any sets A and B with $|A| = |B| = k$, there are $k!$ bijections from A to B.
>
> Let A and B be any two non-empty sets with cardinality $k + 1$. We need to show that there are $(k + 1)!$ bijections from A to B.
>
> Pick an element x in A. (We can do this since A is not empty.)
>
> There are $(k + 1)$ ways to choose x's image $f(x)$. To complete one of these into a bijection of A onto B, we need to make images for the rest of the elements of A. That is, we need to create a bijection from $A - \{x\}$ onto $B - \{f(x)\}$. Both $A - \{x\}$ and $B - \{f(x)\}$ contain k elements. So, by the inductive hypothesis, there are $k!$ bijections from $A - \{x\}$ onto $B - \{f(x)\}$.
>
> So, for each of the $k + 1$ choices for $f(x)$, we have $k!$ ways to complete the whole bijection. This means that we have $(k + 1) \cdot k! = (k + 1)!$ possible bijections from A onto B,which is what we needed to show.

5 A geometrical example

You can also use induction on geometrical objects. For example, *tiling* some area of space with a certain type of puzzle piece means that you can fit the puzzle pieces onto that area of space exactly, with no overlaps or missing areas. A right triomino is a 2-by-2 square minus one of the four squares. (See pictures in Rose pp. 277-278.) I then claim that

Claim 4 *For any positive integer n, a $2^n \times 2^n$ checkerboard with any one square removed can be tiled using right triominoes.*

> Proof: by induction on n.
>
> Base: Suppose $n = 1$. Then our $2^n \times 2^n$ checkerboard with one square remove is exactly one right triomino.
>
> Induction: Suppose that the claim is true for some integer k. That is a $2^k \times 2^k$ checkerboard with any one square removed can be tiled using right triominoes.
>
> Suppose we have a $2^{k+1} \times 2^{k+1}$ checkerboard C with any one square removed. We can divide C into four $2^k \times 2^k$ sub-checkerboards P, Q, R, and S. One of these sub-checkerboards is already missing a square. Suppose without loss of generality that this one is S. Place a single right triomino in the middle of C so it covers one square on each of P, Q, and R.
>
> Now look at the areas remaining to be covered. In each of the sub-checkerboards, exactly one square is missing (S) or already covered (P, Q, and R). So, by our inductive hypothesis, each of these sub-checkerboards minus one square can be tiled with right triominoes. Combining these four tilings with the triomino we put in the middle, we get a tiling for the whole of the larger checkerboard C. This is what we needed to construct.